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STRESS CONCENTRATION AT THE BOUNDARY OF A MICROINHOMOGENEOUS ELASTIC HALF-SPACE

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The perturbation method is used to solve the problem concerning the state of stress of a randomly inhomogeneous half-space under the condition of macro-homogeneous state of stress. Finite formulas for the statistical characteristics of the stresses at the boundary of the half-space are obtained. Problems concerning the state of stress of randomly inhomogeneous media were studied in [1-5] for a plane, a half-plane and a full-space.

An analogous problem in displacements was solved in [6], although no proof was given of the independence of the deformations and stresses on the values assumed by the elastic moduli outside the region occupied by the body the equilibrium of which was under investigation.

1. Let a macrohomogeneous stress-strain state be realized in an inhomogeneous half-space $(x_3 \geqslant 0)$

$$\sigma_{ij}^{(0)} = \langle\sigma_{ij}\rangle, \quad e_{ij}^{(0)} = \langle e_{ij}\rangle \tag{1.1}$$

Here and henceforth the angle brackets denote the operation of mathematical expectation.

We write the stresses and strains in the form

$$\sigma_{ij} = \sigma_{ij}^{(0)} + \sigma_{ij}^{(1)}, \quad e_{ij} = e_{ij}^{(0)} + e_{ij}^{(1)} \tag{1.2}$$

where $\sigma_{ij}^{(1)}$ and $e_{ij}^{(1)}$ represent the fluctuations in the values of stresses and strains.

Hooke's Law and the equations of compatibility of deformations are

$$e_{ij} = S_1\sigma_{ij} - S_2\sigma_{kk}\delta_{ij}; \quad S_1 = \frac{1}{2G}, \quad S_2 = \frac{1}{2G}\frac{\nu}{1+\nu} \tag{1.3}$$

$$\varepsilon_{ijk}\varepsilon_{lmn}e_{kn,\,jm} = 0 \tag{1.4}$$

Here S_i are the pliability moduli related to the shear modulus G and the Poisson's ratio ν, and ε_{ijk} denotes a unit antisymmetric Levi — Civita pseudotensor.

Let us set ($S_k^{(1)}$ denote the fluctuations in the values of pliability moduli)

$$S_k^{(1)} = S_k - S_k^{(0)}, \quad S_k^{(0)} = \langle S_k\rangle \tag{1.5}$$

Substituting (1.2) and (1.3) into (1.4) and using (1.5), we obtain

$$\sigma_{ij,\,kk}^{(1)} + \frac{1}{1+\nu_0}(\sigma_{kk,\,ij}^{(1)} - \sigma_{kk,\,ll}\delta_{ij}) = \frac{1}{S_1^{(0)}}\eta_{ij} \tag{1.6}$$

$$\eta_{ij} = \sigma_{nn}^{(0)}(S_{1,\,ij}^{(1)} - S_{1,\,ll}\delta_{ij}) + \sigma_{ij}^{(0)}S_{1,\,kk}^{(1)} - \sigma_{jk}^{(0)}S_{1,\,ki} - \sigma_{ki}^{(0)}S_{1,\,jk}^{(1)} + \sigma_{kl}^{(0)}S_{1,\,kl}\delta_{ij} + \sigma_{nn}^{(0)}(S_{2,\,ll}^{(1)}\delta_{ij} - S_{2,\,ij}^{(1)})$$

where η_{ij} denotes the incompatibility tensor. In addition, the stresses $\sigma_{ij}^{(1)}$ must satisfy the equilibrium equations and the boundary conditions

$$\sigma_{ij,j}^{(1)} = 0, \quad \sigma_{i3}\,|_{x_3=0} = 0 \ (i = 1, 2, 3) \tag{1.7}$$

Let us write $S_k^{(1)}$ in the form of Fourier integrals

$$S_k^{(1)} = \iiint_{-\infty}^{\infty} f_k(\omega) \exp(i\omega\mathbf{x})\, d\omega \quad (k = 1, 2) \tag{1.8}$$

$$\mathbf{x} = \{x_1, x_2, x_3\}, \quad \omega = \{\omega_1, \omega_2, \omega_3\}$$

where $f_k(\omega)$ is a generalized random function. We seek the solution of the system (1.6), (1.7) in the form

$$\sigma_{ij}^{(1)} = \tau_{ij}^{(1)} + \tau_{ij}^{(2)}$$

where $\tau_{ij}^{(1)}$ is a particular solution of (1.6) and $\tau_{ij}^{(2)}$ is a general solution of the homogeneous system corresponding to (1.6). Using the boundary conditions of (1.7), we obtain the boundary conditions for the solution of the homogeneous system

$$\tau_{3k}^{(2)}\,|_{x_3=0} = -\tau_{3k}^{(1)}\,|_{x_3=0} \tag{1.9}$$

Let us write $\tau_{ij}^{(1)}$ in the form

$$\tau_{ij}^{(1)} = \iiint_{-\infty}^{\infty} \alpha_{ij}^{(k)}(\omega) f_k(\omega) \exp(i\omega\mathbf{x})\, d\omega \quad (k = 1, 2) \tag{1.10}$$

Substituting (1.10) into (1.6) and using (1.8), we obtain

$$\alpha_{ij}^{(k)} = \frac{1}{\varkappa S_1^{(0)}} \frac{1}{\omega^4} [\varkappa\omega^2\eta_{ij}^{(k)} + (\omega_i\omega_j - \omega^2\delta_{ij})\,\eta_{ll}^{(k)}] \quad (k = 1, 2)$$

$$\eta_{ij} = \eta_{ij}^{(1)} + \eta_{ij}^{(2)}, \quad \eta_{ij}^{(2)} = \sigma_{nn}^{(0)} (S_{2,ll}^{(1)}\delta_{ij} - S_{2,ij}^{(1)}), \quad \varkappa = 1 - \nu_0$$

and we can confirm by direct substitution that (1.10) satisfies (1.7).

We seek a solution of the homogeneous system (1.6) identically satisfying (1.7), in the Krutkov's form of [7]

$$\tau_{ij}^{(2)} = -\varepsilon_{ikl}\varepsilon_{jmn}\psi_{ln,km} \quad \left(\psi_{ij} = \frac{1}{1+\varkappa}\gamma\delta_{ij} + \varphi_{ij}\right) \tag{1.11}$$

Here φ_{ij} denote arbitrary harmonic functions, and γ is a particular solution of the equation

$$\frac{\varkappa}{1+\varkappa}\nabla^2\gamma = \varphi_{ij,ij} \tag{1.12}$$

Substituting $\varphi_{11} \equiv \varphi_{22} \equiv \varphi_{33} \equiv \varphi_3, \varphi_{23} \equiv \varphi_2, \ \varphi_{13} \equiv \varphi_1, \ \varphi_{12} \equiv 0$ into the expression within the brackets in (1.11), we find from (1.12)

$$\gamma = \frac{1+\varkappa}{\varkappa} x_3 \varphi_{i,\,i} \quad (i = 1,\, 2) \tag{1.13}$$

Taking into account the condition that the stresses $\tau_{ij}{}^{(2)}$ vanish at infinity, we write the harmonic functions φ_k $(k = 1, 2, 3)$ in the form

$$\varphi_k = \iint\limits_{-\infty}^{\infty} a_k^{(n)}(\boldsymbol{\omega}) f_n(\boldsymbol{\omega}) \exp(i\boldsymbol{\omega}_* \mathbf{x}_* - \omega_* x_3)\, d\boldsymbol{\omega}_* \quad (n = 1,\, 2) \tag{1.14}$$

$$\mathbf{x}_* = \{x_1,\, x_2\}, \quad \boldsymbol{\omega}_* = \{\omega_1,\, \omega_2\}$$

Using now (1.10), (1.14), (1.13) and (1.11), we obtain the following final expressions for the stress perturbations:

$$\sigma_{ij}^{(1)} = \iiint\limits_{-\infty}^{\infty} [\alpha_{ij}^{(k)} f_k \exp(i\omega_3 x_3) + \Omega_{ij}^{(k)} f_k \exp(-\omega_* x_3)] \exp(i\omega_* x_*)\, d\boldsymbol{\omega} \tag{1.15}$$

$$\Omega_{ij}^{k} = i\Big[\frac{1}{\varkappa}(\omega_i\omega_j x_3 - 2\delta_{ij}\omega_*)\,\omega_m a_m^{(k)} + \omega_*(2\delta_{ij}\omega_m a_m^{(k)} -$$

$$\omega_i a_j^{(k)} - \omega_j a_i^{(k)}) - i\omega_i\omega_j a_3^{(k)}\Big]$$

$$\Omega_{l3}^{(k)} = \frac{1}{\varkappa}\omega_l(1 - \omega_* x_3)\,\omega_m a_m^{(k)} + \omega_*{}^2 a_l^{(k)} - \omega_l\omega_m a_m^{(k)} + i\omega_l\omega_* a_3^{(k)}$$

$$\Omega_{33}^{(k)} = i\Big[-\frac{1}{\varkappa}\omega_*{}^2 x_3 \omega_m a_m^{(k)} + i\omega_*{}^2 a_3^{(k)}\Big]$$

$$(i,\, j = 1,\, 2,\, 3; \quad k,\, l,\, m = 1,\, 2)$$

where $a_m{}^{(k)}$ $(k = 1, 2;\ m = 1, 2, 3)$ are obtained from the boundary conditions (1.9)

$$a_1^{(k)} = (1-\varkappa)\frac{\omega_1\omega_2}{\omega_*{}^4}\alpha_{23}^{(k)} + b_1, \quad a_2^{(k)} = (1-\varkappa)\frac{\omega_1\omega_2}{\omega_*{}^4}\alpha_{13}^{(k)} + b_2 \tag{1.16}$$

$$a_3^{(k)} = \frac{1}{\omega_*{}^2}\alpha_{33}^{(k)}$$

$$b_n = i(1-\varkappa)\frac{\omega_n(\omega_*{}^2 - \omega_n{}^2)}{\omega_*{}^5}\alpha_{33}^{(k)} - \frac{\omega_*{}^2 - (1-\varkappa)\,\omega_n{}^2}{\omega_*{}^5}(\omega_*\alpha_{n3}^{(k)} + i\omega_n\alpha_{33}^{(k)})$$

The formulas (1.15) contain Fourier transforms f_k of the functions $S_k{}^{(1)}$ $(k = 1, 2)$ and this seems to imply, at first sight, the necessity of defining these functions on the whole space in order to determine the stresses in the half-space. We shall show that this is not so, i.e. that $S_k{}^{(1)}$ at $x_3 < 0$ do not affect the values of the stresses at $x_3 \geqslant 0$. We have for the functions f_k

$$f_k(\boldsymbol{\omega}) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} F_k(\omega_1,\, \omega_2,\, u_3)\exp(-i\omega_3 u_3)\, du_3 \quad (k = 1,\, 2) \tag{1.17}$$

$$F_k(\omega_1,\, \omega_2,\, u_3) = \frac{1}{(2\pi)^2}\iint\limits_{-\infty}^{\infty} S_k^{(1)}(u)\exp(-i\omega_* u_*)\, du_*$$

Using (1.17) we can reduce the Fourier transform in x_1, x_2 of the stresses $\sigma_{lm}^{(1)}(x_1, x_2, x_3)$ in (1.15) to the form

$$P_{lm}(\omega_1, \omega_2, x_3) = \int_{-\infty}^{\infty} G_{lm}(\omega_1, \omega_2, x_3, u_3)\, du_3 \tag{1.18}$$

$$G_{lm}(\omega_1, \omega_2, x_3, u_3) = \frac{1}{2\pi} F_k(\omega_1, \omega_2, u_3) \int_{-\infty}^{\infty} \{\alpha_{lm}^{(k)} \exp(i\omega_3 x_3) + A_{lm}^{(k)} \exp(-i\omega_* x_3)\} \exp(-i\omega_3 u_3)\, d\omega_3$$

$$A_{lm}^{(k)} = i\frac{1}{\varkappa}(\omega_l \omega_m x_3 - 2\omega_* \delta_{lm})\, \omega_p a_p^{(k)} + i(2\delta_{lm}\omega_p a_p^{(k)} - \omega_l a_m^{(k)} - \omega_m a_l^{(k)})\, \omega_* + \omega_l \omega_m a_3^{(k)} \quad (l, m, p, k = 1, 2)$$

Direct computation of the integral in (1.18) shows that $G_{lm}(\omega_1, \omega_2, x_3, u_3) = 0$ when $u_3 < 0$, and this proves that $P_{lm}(\omega_1, \omega_2, x_3)$ is independent of $S_k^{(1)}(\mathbf{u})$ when $u_3 < 0$. The calculations are identical for the remaining three stresses.

2. We assume that the random fields $S_k^{(1)}$ $(k = 1, 2)$ are statistically homogeneous and isotropic, connected by statistically homogeneous and isotropic relations, and have the following known correlation functions:

$$K_{lm}(\xi) = \langle \overline{S_l^{(1)}(\mathbf{x})}\, S_m^{(1)}(\mathbf{x} + \xi) \rangle$$

and write the components of the correlation stress tensor

$$K_{pqst}(\mathbf{x}, \mathbf{x}') = \langle \overline{\sigma_{pq}(\mathbf{x})}\, \sigma_{st}(\mathbf{x}') \rangle$$

in the following form (where a prime denotes complex conjugate quantities):

$$K_{pqst}(\mathbf{x}, \mathbf{x}') = \iiint_{-\infty}^{\infty} \{\alpha_{pq}^{(m)} \alpha_{st}^{(n)} \Phi_{mn}(\omega) \exp[i\omega_3(x_3' - x_3)] - A_{pq}^{(k)} \alpha_{st}^{(l)} \Phi_{kl}(\omega) \exp(i\omega_3 x_3' - \omega_* x_3) + B_{st}^{(r)} \alpha_{pq}^{(j)} \Phi_{rj}(\omega) \exp \times (-\omega_* x_3' - i\omega_3 x_3) + A_{pq}^{(\nu)} B_{st}^{(\mu)} \Phi_{\nu\mu}(\omega) \exp[-\omega_*(x_3' + x_3)]\} \times \exp[i\omega_*(\mathbf{x}_*' - \mathbf{x}_*)]\, d\omega \tag{2.1}$$

$$\Phi_{lm}(\omega) = \Phi_{lm}(\omega) = \frac{1}{(2\pi)^{3/2}} \int_0^{\infty} K_{lm}(\xi) J_{1/2}(\omega\xi)(\omega\xi)^{-1/2} \xi^2\, d\xi$$

$$\omega^2 = \omega_k \omega_k, \quad \xi^2 = \xi_k \xi_k, \quad p, q, s, t = 1, 2$$

(see [8]). $B_{st}^{(l)}$ are obtained from $A_{pq}^{(k)}$ (see (1.18)) by replacing p, q, and k by s, t and l; x_3 by x_3' and $a_3^{(k)}$ by $-a_3^{(l)}$. The remaining components of the correlation tensor can be obtained in the same way.

We see from (2.1) that the stress field is stationary along the axes x_1 and x_2, and nonstationary in the direction of x_3, i.e.

$$K_{pqst}(\mathbf{x}, \mathbf{x}') = K_{pqst}(\mathbf{x}_*' - \mathbf{x}_*, x_3', x_3)$$

Let us inspect the stresses $\sigma_{ij}^{(1)}$ $(i, j = 1, 2)$ at the boundary of the half-space $x_3 = 0$. We set $\nu = \nu_0 = \mathrm{const}$ in the formulas connecting S_k $(k = 1, 2)$ with the Poisson's ratio ν and shear modulus G. Then $S_2(\mathbf{x}) = \nu_0(1 + \nu_0)^{-1}$ $S_1(\mathbf{x}) = S(\mathbf{x})$. Let us again limit ourselves to the case when $\sigma_{11}^{(0)} = \sigma_{22}^{(0)} = p$, $\sigma_{12}^{(0)} = \sigma_{13}^{(0)} = \sigma_{23}^{(0)} = \sigma_{33}^{(0)} = 0$. The expressions for the correlation functions (2.1) at the boundary of the half-space will have the following form:

$$K_{pppp}(\xi) = \frac{\mu^2}{\varkappa^2}\iiint_{-\infty}^{\infty} \frac{1}{\omega^4\omega_*^2}\{4\varkappa^2(\omega_*^2 - \omega_p^2)[(\varkappa - 1)\omega_*^2 - \omega_p^2] \times (\varkappa\omega^2 - \omega_*^2) + \omega_*^2[\varkappa(\omega^2 + \omega_*^2 - \omega_p^2) - \omega_*^2]^2\}\Phi(\omega)\exp(i\omega_*\xi_*)d\omega \tag{2.2}$$

$$K_{1122}(\xi) = \frac{\mu^2}{\varkappa^2}\iiint_{-\infty}^{\infty} \frac{1}{\omega^4\omega_*^2}\{2\varkappa^2(\varkappa\omega^2 - \omega_*^2)(2\varkappa\omega_1^2\omega_2^2 - \omega_*^4) + 2i\varkappa^2\omega_*\omega_3(\varkappa\omega^2 - \omega_*^2)(\omega_1^2 - \omega_2^2) + \omega_*^2[\varkappa(\omega^2 + \omega_1^2) - \omega_*^2] \times [\varkappa(\omega^2 + \omega_2^2) - \omega_*^2]\}\Phi(\omega)\exp(i\omega_*\xi_*)d\omega$$

$$K_{1212}(\xi) = \frac{\mu^2}{\varkappa^2}\iiint_{-\infty}^{\infty} \varkappa^2\frac{\omega_1^2\omega_2^2}{\omega^4\omega_*^2}[4\varkappa^2\omega^2 - (4\varkappa - 1)\omega_*^2]\Phi(\omega)\exp(i\omega_*\xi_*)d\omega$$

$$p = 1, 2, \quad \mu^2 = \frac{p^2}{S_0^2}, \quad S_0 = \langle S(x)\rangle$$

Passing in (2.2) to the spherical coordinates and integrating over the angles we obtain, e.g. for K_{1212} and K_{1211}

$$K_{1212}(\xi_*) = 2\mu^2\pi^2\frac{\partial^3}{\partial\xi_1\partial\xi_2^2}\int_0^\infty \Big[(1 - 4\varkappa)\frac{1}{\omega^2}\frac{\partial}{\partial\xi_1}J_{-1/2}(\Omega)J_{1/2}(\Omega) - 4\varkappa^2\frac{1}{\omega}\frac{\xi_1}{\xi_*}J_{1/2}^2(\Omega)\Big]\Phi(\omega)d\omega$$

$$K_{1211}(\xi_*) = \mu^2\pi^2\frac{\partial^2}{\partial\xi_1\partial\xi_2}\int_0^\infty \Big\{\Big[2(4\varkappa - 1)\frac{\partial^2}{\partial\xi_2^2}\frac{1}{\omega^2} + 5 - 4\varkappa - \frac{3}{2}\frac{1}{\varkappa}\Big]J_{-1/2}(\Omega)J_{1/2}(\Omega) + \Big(1 - \frac{1}{2}\frac{1}{\varkappa}\Big)J_{-3/2}(\Omega)J_{3/2}(\Omega) + 8\varkappa^2\frac{\partial}{\partial\xi_2}\frac{1}{\omega}\frac{\xi_2}{\xi_*}J_{1/2}^2(\Omega)\Big\}\Phi(\omega)d\omega, \qquad \Omega = \frac{\omega\xi_*}{2}$$

The remaining components of the correlation tensor have the same form, but are not given here because of their length.

Assuming $\xi_* = 0$, we obtain the following formulas for the dispersions:

$$D_{1111} = \frac{\mu^2}{\varkappa^2}\frac{4\pi}{15}(15\varkappa^4 - 32\varkappa^3 + 44\varkappa^2 - 28\varkappa + 8)D \tag{2.3}$$

$$D_{1122} = \frac{\mu^2}{\varkappa^2}\frac{4\pi}{15}(5\varkappa^4 - 24\varkappa^3 + 42\varkappa^2 - 28\varkappa + 8)D$$

$$D_{2222} = D_{1111}, \quad D_{1211} = D_{1222} = 0$$

$$D_{1212} = \frac{\mu^2}{\varkappa^2}\frac{4\pi}{15}(5\varkappa^4 - 4\varkappa^3 + \varkappa^2)D$$

$$D = \int_0^\infty \omega^2 \Phi(\omega)\, d\omega, \quad \omega^2 = \omega_m \omega_m$$

Let us write (2.3) in the form $D_{ijkl} = \mu^2 D v^2$ $(i, j, k, l = 1,2)$, where v denotes the so-called variability coefficient. Computing v for the case $\nu_0 = 0.25$ yields the following result:

$$i = j = k = l = 1 \text{ and } i = j = k = l = 2, \quad v^2 = 2.76$$
$$i = j = 1;\ k = l = 2, \quad v^2 = 2.96$$
$$i = k = 1;\ j = l = 2, \quad v^2 = 0.675$$

and the corresponding values of the variability coefficient for the space are (*) 2.51, 1.69 and 0.322.

The above computations show that the increase in dispersion (especially in D_{1122} and D_{1212}) caused by the boundary is substantial, and must undoubtedly be taken into account in practical computations.

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*) Podalkov V. V. and Romanov V. A. Statistical characteristics of microinhomogeneous space. Papers presented at the IV-th All-Union Conference on the Problems of Reliability in Constructional Mechanics. Vil'nius, 1975.